

Transient growth in linearly stable gravity-driven flow in porous media

G.J.M. Pieters *, C.J. van Duijn

Department of Mathematics and Computer Science, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands

Received 26 January 2005; accepted 25 April 2005

Available online 13 June 2005

Abstract

In this paper we study gravitational instability of a saline boundary layer formed by evaporation induced upward throughflow at the horizontal surface of a porous medium. Van Duijn et al. [P.A.C. Raats, D. Smiles, and A.W. Warrick (Eds.), Environmental Mechanics – Water, Mass and Energy Transfer in the Biosphere – The Philip Volume, Geophys. Monographs, Vol. 129, American Geophysical Union, 2002, pp. 155–169], derived stability bounds by means of linear stability analysis and an (improved) energy method. These bounds do not coincide, i.e. there exists a subcritical region or stability gap in the system parameter space which is due to the asymmetry of the linear part of the perturbation equations. We show that the linear operator can be symmetrized by means of a similarity transformation. For system parameter values in the stability gap, we show that there exist initial perturbations for which the linearly stable system exhibits transient growth. We show that transient growth is norm dependent by considering weighted norms, which are induced by a one-parameter family of similarity transformations.

© 2005 Elsevier SAS. All rights reserved.

Keywords: Porous media flow; Gravitational instability; Linear stability analysis; Transient growth

1. Introduction

In this paper we investigate the dynamics of groundwater beneath salt lakes. In particular we study the stability of an equilibrium boundary layer formed by evaporation induced upward throughflow at the horizontal surface of a porous medium. Central issue is to quantify the boundary layer stability or its gravitational instability in terms of the system parameters. This problem was first introduced by [1,2]. It was later further detailed in [3] where the method of linearised stability and the energy method (with different constraints) were used to establish stability criteria for the boundary layer. Starting point for both methods is a nonlinear perturbation equation which has the abstract form

$$\left. \begin{aligned} \frac{ds}{dt} &= \mathcal{L}s + \mathcal{N}(s) \\ s(t) &\in \mathcal{H} \end{aligned} \right\} \quad \text{for } t > 0, \quad (1)$$

$s(0) = f^*,$

* Corresponding author.

E-mail addresses: g.j.m.pieters@tue.nl, gjpieters@yahoo.com (G.J.M. Pieters).

where s denotes the perturbation with respect to some ground state solution, and where \mathcal{L} represents a linear and \mathcal{N} a nonlinear operator. The boundary conditions are grouped in the solution space \mathcal{H} to which s is said to belong. The energy method uses both \mathcal{L} and \mathcal{N} , whereas linear stability analysis is applied to the operator \mathcal{L} only. Both approaches give stability bounds in terms of the system parameters.

The stability bounds found by [3] do not coincide, i.e. there exists a stability gap which was shown to depend on the constraints used in the energy method. One of the first observations regarding the occurrence of stability gaps was given by [4–6], and, independently and in a different context, by [7,8] and later by [9]. For a general overview we refer to [10]. These authors showed that one of the essential connections between linear and nonlinear theory is the idea of symmetry of the *linear* operator \mathcal{L} . Therefore we write $\mathcal{L} = \mathcal{L}_s + \mathcal{L}_a$, where \mathcal{L}_s denotes the symmetric and \mathcal{L}_a the asymmetric part of \mathcal{L} . In particular, when the nonlinear term satisfies $(\mathcal{N}(s), s) \leq 0$, where (\cdot, \cdot) denotes an appropriately chosen inner product, usually the L^2 -inner product, [9] showed that the nonlinear stability analysis reduces to the study of the spectrum of the *symmetric* part of the *linear* operator. Hence, when \mathcal{L} is symmetric, both methods give similar stability bounds.

The presence of throughflow in the salt lake problem results in a first-order term in the operator \mathcal{L} , implying $\mathcal{L}_a \neq 0$. In addition, the nonlinear term satisfies $(\mathcal{N}(s), s) \equiv 0$ and hence the theory as developed in [9] is applicable for this case: The spectrum of \mathcal{L} does not coincide with the spectrum of \mathcal{L}_s and this results in a stability gap.

To quantify this gap and to understand the behaviour of the physical system in this – so-called – subcritical region, a modified energy method is needed [9]. One method is to formulate a ‘generalized energy’ by coupling different norms with suitable coupling parameters, see [11]. The idea is to optimize these coupling parameters to obtain a sharper energy stability bound. In some cases it even removes the stability gap. Such energies are introduced on purely heuristic grounds.

An alternative approach, which is followed in this paper, is to use the concept of symmetrizable operators. By this we mean that \mathcal{L} may be asymmetric with respect to one inner product, but symmetric with respect to another. This can usually be achieved by reformulating the original problem in terms of a newly defined quantity. For example, such a new quantity may be equal to the original quantity multiplied by a weight function. A sufficient condition for which an operator \mathcal{L} is symmetrizable, is that there exists an invertible operator \mathcal{M} such that $\mathcal{M}^{-1}\mathcal{L}\mathcal{M}$ is symmetric. Such an operator defines a *similarity transformation* and the resulting operator $\mathcal{M}^{-1}\mathcal{L}\mathcal{M}$ is called *similar* (to \mathcal{L}). Besides being symmetric, a similar operator has the important property that its spectrum is a subset of the spectrum of \mathcal{L} . Once we have found such an operator \mathcal{M} , we can apply [9]: due to the symmetry of the similar operator, the linear stability bound coincides with the nonlinear stability bound provided that the nonlinear term satisfies $(\mathcal{N}(s), s)_{\mathcal{M}} \leq 0$, where $(\cdot, \cdot)_{\mathcal{M}} := (\mathcal{M}^{-1}(\cdot), \mathcal{M}^{-1}(\cdot))$. Here nonlinear stability is measured in the energy norm induced by $(\cdot, \cdot)_{\mathcal{M}}$. For other applications of similarity transformations we refer to [10, Sections 4 and 5]. The use of transformations and the consequences for the stability analysis is also discussed in [12].

The introduction of different inner-products affects the orthogonality of the *eigenfunctions* of the (asymmetric) operator \mathcal{L} : for instance, the eigenfunctions may be non-orthogonal with respect to the L^2 inner-product, but orthogonal with respect to a weighted L^2 inner-product. This results from the fact that the operator is *nonnormal* [13]. Due to the non-orthogonality of the eigenfunctions, nonnormal operators exhibit particular transient behaviour which cannot be captured by linear stability analysis. In other words, the linearly stable but nonnormal system may temporarily move far away from equilibrium before approaching it as $t \rightarrow \infty$, while undergoing a considerable (transient) algebraic growth of the energy norm. This is studied by [14] and later by [15]. The concept is that a group of eigenfunctions are nearly linearly dependent so that particular initial disturbances may involve large coefficients.

With this mechanism in mind, one could determine the optimal initial condition that produces the largest relative energy growth during a certain time period. This is done by [14]. Their variational optimization method relies on the fact that every initial perturbation can be decomposed into eigenfunctions, i.e. the eigenfunctions form a complete set. For certain problems on semi-infinite domains, however, like the Blasius boundary layer [16,17], one can show that there is a finite number of eigenfunctions corresponding to a discrete point spectrum and that there are solutions that correspond to an uncountable point spectrum. The latter is sometimes referred to as the continuum. In this paper we consider a saline boundary layer in a semi-infinite domain for which both spectra occur. It is clear the one can only determine a finite number of discrete eigenvalues and corresponding eigenfunctions. Hence the determination of the optimal initial perturbation via a variational optimization procedure is limited since one should have to incorporate the remaining part of the spectrum, see also [18].

However, it is still possible to find initial perturbations that initiate transient growth. These special perturbations can be determined by analysing the *numerical range* [13] of the operator \mathcal{L} . The numerical range is the largest eigenvalue of the operator \mathcal{L}_s and this eigenvalue is equal to the initial slope of the energy norm [15].

This interpretation has led to general analysis of nonnormal systems that has since been used extensively to understand transient growth in deformation and shear flows. For further references see, for example, [19] and [20].

This paper is organized as follows. In Section 2, we formulate the porous medium flow problem and we introduce the equilibrium saline boundary layer as the ground state solution.

In Section 3 we analyse the spectrum of the linear operator; i.e. we study the linearised perturbation equations. We also recall some earlier results [3] about the stability bounds: a stability bound resulting from the linear theory and a smaller bound resulting from an energy method. Below the latter bound all perturbations decay in L^2 -sense with respect to time.

In Sections 4 and 5 we consider the behaviour of the linearised perturbation equations for parameter values between these stability bounds. In Section 4 we demonstrate transient growth of the L^2 -norm of perturbations and we construct some bounds on this growth. In Section 5 we show that transient growth is norm dependent. In particular we introduce a one-parameter family of transformations that imply weighted L^2 -norms. Depending on the parameter value, transient growth may occur or disappear.

Conclusions are summarized in Section 6.

2. Problem formulation

Following [3], we consider a homogeneous and uniform isotropic porous medium occupying the three-dimensional halfspace

$$\Omega = \{(x, y, z) : (x, y) \in \mathbb{R}^2, z > 0\}.$$

The dimensionless equations in terms of the Boussinesq approximation are given by [21,1]:

- Fluid incompressibility

$$\nabla \cdot \mathbf{U} = 0; \quad (2a)$$

- Darcy's law

$$\mathbf{U} + \nabla P - S \mathbf{e}_z = \mathbf{0}; \quad (2b)$$

- Salt transport

$$\partial_t S + R_s \mathbf{U} \cdot \nabla S = \nabla^2 S \quad (2c)$$

in Ω and for all $t > 0$, subject to the boundary conditions

$$\mathbf{U} \cdot \mathbf{e}_z = -R_s^{-1} \quad \text{and} \quad S = 1 \quad \text{at } z = 0, \text{ for all } t > 0. \quad (3)$$

Here S denotes salt saturation, \mathbf{U} fluid velocity and P an appropriately chosen dimensionless pressure. Further, \mathbf{e}_z denotes the unit vector in z -direction, pointing downwards. The system Rayleigh number R_s is given by

$$R_s = \frac{(\rho_m - \rho_r)g\kappa}{\mu E}, \quad (4)$$

where μ is fluid viscosity, κ medium permeability, g gravity constant, E evaporation rate, ρ_m maximum fluid density at the outflow boundary, and ρ_r fluid density in 'natural circumstances' (i.e. far away from the outflow boundary).

To obtain a unique solution satisfying Eqs. (2) and boundary conditions (3), an initial condition is needed:

$$S|_{t=0} = f^* \quad \text{in } \Omega, \quad (5)$$

where f^* denotes the – as yet – unspecified salt saturation at the initial time $t = 0$.

Remark 1. Note that we do not impose boundary conditions at $z = \infty$. The behaviour at $z = \infty$ is implied by the initial condition f^* and by physically plausible growth conditions for the solutions. This is explained in [22].

2.1. The ground state solution

In this paper we are concerned with the stability of the time-independent solution of (2) and (3) that vanishes at large depth, i.e. as $z \rightarrow \infty$. This solution is called the ground state. It represents an equilibrium boundary layer below the surface $z = 0$, which is sustained by evaporation induced throughflow. It is explicitly given by the uniform upflow

$$\mathbf{U} = \mathbf{U}_0 := -R_s^{-1} \mathbf{e}_z \quad \text{in } \Omega, \quad (6)$$

and by the salt saturation $S = S_0(z)$ satisfying

$$\begin{cases} \frac{d^2 S_0}{dz^2} + \frac{dS_0}{dz} = 0, & \text{for } z > 0, \\ S_0(0) = 1, & \lim_{z \rightarrow \infty} S_0(z) = 0. \end{cases}$$

Clearly,

$$S_0(z) = e^{-z}, \quad z > 0. \quad (7)$$

The corresponding ground state pressure $P = P_0$ is found by integrating Darcy's law (2b).

2.2. Perturbation equations

The stability analysis is based on the expansion

$$S = S_0 + s, \quad \mathbf{U} = \mathbf{U}_0 + \mathbf{u}, \quad P = P_0 + p, \quad (8)$$

where $\mathbf{u} = (u, v, w)$. Substituting (8) into Eqs. (2) and writing R instead of R_s , yields for the perturbations the system (in Ω and for all $t > 0$)

$$\nabla \cdot \mathbf{u} = 0, \quad (9a)$$

$$\mathbf{u} + \nabla p - s\mathbf{e}_z = \mathbf{0}, \quad (9b)$$

$$\partial_t s - \partial_z s - R e^{-z} w + R \mathbf{u} \cdot \nabla s = \nabla^2 s, \quad (9c)$$

and the homogeneous boundary conditions

$$s = \mathbf{u} \cdot \mathbf{e}_z = w = 0 \quad \text{at } z = 0, \text{ for all } t > 0. \quad (10)$$

As in [23] we note that Eqs. (9a) and (9b) can be combined to give for s and w the linear relation

$$\nabla^2 w = \nabla_{\perp}^2 s \quad \text{in } \Omega. \quad (11)$$

Here ∇_{\perp}^2 denotes the horizontal Laplacian $\partial_{xx} + \partial_{yy}$.

3. Linear stability of the equilibrium boundary layer

The stability of the ground state (6), (7) was investigated by [3]. They used linear stability theory, i.e. disregarding the product term $R \mathbf{u} \cdot \nabla s$ in Eq. (9c), and an improved energy method using Eq. (11) as pointwise constraint. Restricted to periodic perturbations of the form

$$\{s, w\}(x, y, z, t) = \{s, w\}(z, t) e^{i(a_x x + a_y y)}, \quad (12)$$

where a_x and a_y denote the horizontal wavenumbers, they obtained two stability bounds. With $a^2 = a_x^2 + a_y^2$, these bounds are represented in the (a, R) -plane by the curves

$$0 < R_E(a) < R_L(a) \quad \text{for } a > 0. \quad (13)$$

The ground state is unstable above the curve $R_L(a)$, and definitely stable below the curve $R_E(a)$ in the sense that the L^2 -norm of any periodic perturbation decays in time when $0 < a < \infty$ and $0 < R < R_E(a)$. The two bounds are shown in Fig. 1.

The main goal of this paper is to understand the behaviour of (x, y) -periodic solutions of the linearized perturbation equations for points (a, R) in between these stability bounds. We start with some observations concerning the spectrum and the corresponding eigenfunctions.

3.1. Properties of the spectrum of the linear operator

We consider the linear part of (9c) and (11) in $Q := \{(z, t) : z > 0, t > 0\}$. Assuming $f^* = f^*(z)$ and periodicity of s and w with respect to the horizontal plane, as in (12), we obtain the equations

$$(LP) \quad \begin{cases} \partial_t s = D^2 s + D s - a^2 s + R e^{-z} B(s) =: L s & \text{in } Q, \\ B^{-1}(w) := -a^{-2} D^2 w + w = s & \text{in } \mathbb{R}_+ \text{ and for each } t > 0, \\ w(0, t) = s(0, t) = 0 & \text{for all } t > 0, \\ s(z, 0) = f(z) & \text{for all } z > 0. \end{cases}$$

In (LP) we used the notation $D := \partial/\partial z$ or d/dz . Because s is the perturbation of (7) we have $f(z) = f^*(z) - e^{-z}$. Note that we do not impose the boundary condition $\lim_{z \rightarrow \infty} w(z) = 0$ to solve the w -problem. This is a consequence of the fact that

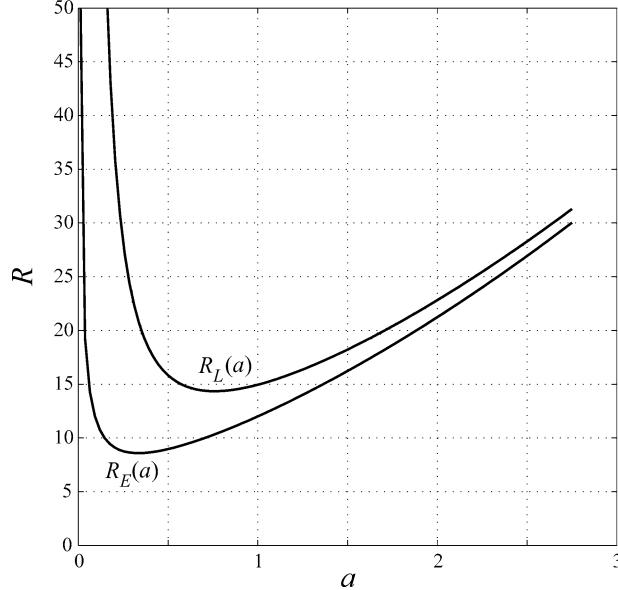


Fig. 1. Stability bounds involving the lowest eigenvalue $R = R_{\min}$ versus wavenumber a : top curve $R = R_L(a)$ results from the linear stability method, bottom curve $R = R_E(a)$ results from the energy method.

$s(\cdot, t) \in L^2(\mathbb{R}_+)$, see [22] for details. To investigate the spectrum of L , we fix $a > 0$ and $R > 0$ and consider the eigenvalue problem

$$Ls = D^2s + Ds - a^2s + R e^{-z} B(s) = \sigma s, \quad s \in H_0^1(\mathbb{R}_+). \quad (14)$$

Here $H_0^1(\mathbb{R}_+)$ denotes the Hilbert space of square integrable functions having a square integrable derivative and vanishing at $z = 0$. Since no explicit solutions of (14) are known, we solved the eigenvalue problem numerically by means of a modified Chebyshev–Galerkin method. Details of this method are discussed in [24]. Later on in this section we show by means of a similarity transformation that the discrete eigenvalues $\{\sigma_i = \sigma_i(a, R)\}$ of (14) are real. Let them be ordered according to

$$\cdots < \sigma_n < \cdots < \sigma_2 < \sigma_1 =: \sigma_{\max}.$$

For a and R in relevant ranges, Fig. 2(A) shows numerically obtained isocurves of σ_{\max} . Putting $R = 0$ in (14) gives the explicit expression $\sigma_{\max} = -a^2 - \frac{1}{4}$. This is recovered numerically. Further observe that the isocurves become vertical as $R \searrow 0$. The dashed isocurve (corresponding to $\sigma_{\max} \equiv 0$) in Fig. 2(A) is usually referred to as the neutral stability curve. It is often found by solving the rewritten eigenvalue problem

$$\mathcal{L}_\sigma s := D^2s + Ds - (a^2 + \sigma)s = -R e^{-z} B(s) =: R \mathcal{B} s, \quad (15)$$

with $s \in H_0^1(\mathbb{R}_+)$, see [3]. In other words, the role of σ and R as eigenvalues can be interchanged. Suppose again that the set of eigenvalues $\{R_i = R_i(a, \sigma)\}$ of problem (15) are ordered by $R_{\min} := R_1 < R_2 < \cdots < R_n < \cdots$. Then the neutral stability curve is defined by $R_L(a) := R_{\min}(a, \sigma = 0)$. It is also shown as the top curve in Fig. 1.

In the sequel of this section we investigate the behaviour of $R_{\min}(a, \sigma)$ near $\sigma = 0$. Fig. 2(A) suggests that for fixed $a > 0$, at least for σ sufficiently close to zero, we have $\sigma \geq 0$ if and only if $R \geq R_{\min}(a, \sigma = 0) = R_L(a)$. This property is crucial in linearised stability theory. Its physical interpretation is that the conductive ground state exchanges stability with a convective flow regime. In general, exchange of stability is straightforward to prove when the linear operator L is symmetric with respect to $L^2(\mathbb{R}_+)$, see [9]. However, the operator L (and \mathcal{L}_σ) is asymmetric in $L^2(\mathbb{R}_+)$ due to the occurrence of the first order derivative. To show the stability exchange, we will make use of a particular (similarity) transformation which symmetrizes the operator L in $L^2(\mathbb{R}_+)$. Following [25, Section 5], we set

$$s(z, t) = M u(z, t) := e^{-\frac{1}{2}z} u(z, t). \quad (16)$$

Then

$$Ls = e^{-\frac{1}{2}z} \left[D^2u + \left(-\frac{1}{4} - a^2 \right) u + R e^{-\frac{1}{2}z} B(e^{-\frac{1}{2}z} u) \right] =: M \hat{L} M^{-1} s, \quad (17)$$

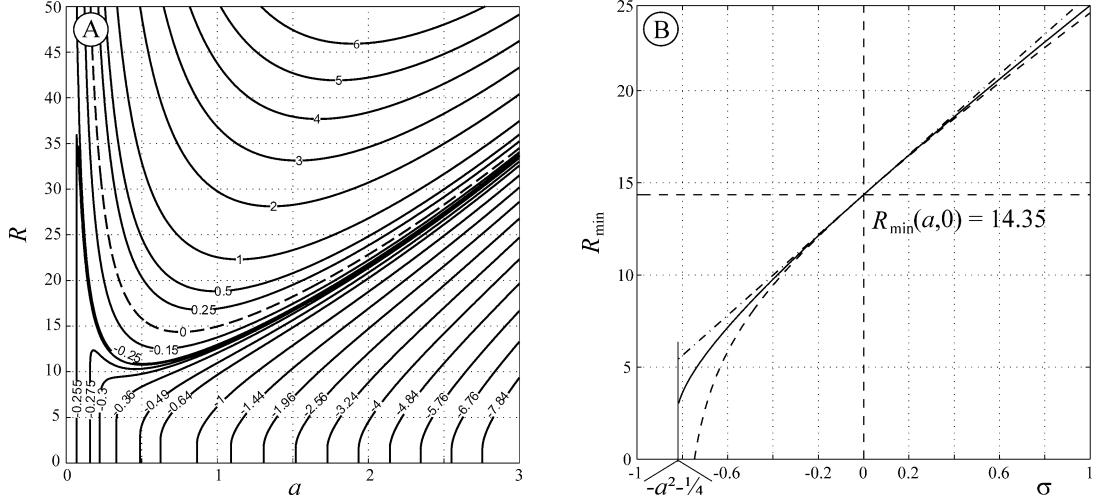


Fig. 2. (A) Isocurves of the largest approximate eigenvalue σ_{\max} of operator L . The dashed curve $\sigma_{\max} \equiv 0$ corresponds to the neutral stability curve $R_L(a)$ (see text for its definition). (B) Upper and lower bounds of $R_{\min}(a, \sigma)$ for $a = 0.759$ as function of the growthrate σ .

where \hat{L} is given by

$$\hat{L} = D^2 + \left(-\frac{1}{4} - a^2\right)I + Re^{-\frac{1}{2}z}B(e^{-\frac{1}{2}z}\cdot), \quad (18)$$

in which I denotes the identity operator. Now we consider the problem

$$(\widehat{LP}) \quad \begin{cases} \partial_t u = D^2 u + \left(-\frac{1}{4} - a^2\right)u + Re^{-\frac{1}{2}z}B(e^{-\frac{1}{2}z}u) = \hat{L}u, \\ B^{-1}(w) = -a^{-2}D^2w + w = e^{-\frac{1}{2}z}u, \end{cases}$$

in Q , with

$$w(0, t) = u(0, t) = 0 \quad \text{for all } t > 0 \quad (19)$$

and

$$u(z, 0) = g(z) := e^{\frac{1}{2}z}f(z) \quad \text{for all } z > 0. \quad (20)$$

Note that the operator \hat{L} is symmetric with respect to $L^2(\mathbb{R}_+)$. Clearly, the spectrum of \hat{L} is contained in the spectrum of L . In fact, the part of the spectrum of L for which the eigenfunctions belong to $\{s \in L^2(\mathbb{R}_+): se^{\frac{1}{2}z} \in L^2(\mathbb{R}_+)\}$ coincides with the spectrum of \hat{L} with respect to $L^2(\mathbb{R}_+)$. We are now in a position to prove exchange of stability quite easily by using \hat{L} instead of L . For this purpose we rewrite the eigenvalue problem $\hat{L}u = \sigma u$, as in (15), in the form

$$\hat{L}_\sigma u := D^2 u - \left(a^2 + \frac{1}{4} + \sigma\right)u = R\hat{\mathcal{B}}u, \quad (21)$$

with $\hat{\mathcal{B}}u := -e^{-\frac{1}{2}z}B(e^{-\frac{1}{2}z}u)$. We prove

Theorem 2. *Let the smallest eigenvalue of (21) be denoted by $R_{\min}(a, \sigma)$ and suppose that it depends smoothly on $a > 0$ and $\sigma > -a^2 - \frac{1}{4}$. Then there exist a positive constant c and a smooth function $h: (-a^2 - \frac{1}{4}, \infty) \mapsto \mathbb{R}$, satisfying $h(0) = 0$, $h(\sigma) \geq 0$ if $\sigma \geq 0$ and $h(\sigma) \leq c\sigma$ for all $\sigma > -a^2 - \frac{1}{4}$, such that*

$$h(\sigma) \leq R_{\min}(a, \sigma) - R_L(a) \leq c\sigma. \quad (22)$$

Moreover, $h''(0) < 0$.

Proof. First observe that

$$(\hat{\mathcal{B}}u, u) = -(B(e^{-\frac{1}{2}z}u), e^{-\frac{1}{2}z}u) = -(w, s),$$

and $(w, s) = a^{-2} \|Dw\|_2^2 + \|w\|_2^2 > 0$. Here (\cdot, \cdot) denotes the usual $L^2(\mathbb{R}_+)$ inner product. Hence $(\hat{\mathcal{B}}u, u) < 0$. Further, $\hat{\mathcal{L}}_\sigma = \hat{\mathcal{L}}_0 - \sigma I$ where I denotes the identity operator. Since $\hat{\mathcal{L}}_\sigma$ and $\hat{\mathcal{B}}$ are self-adjoint, we have the following variational characterization of $R_{\min}(a, \sigma)$:

$$0 < R_{\min}(a, \sigma) = \inf_{\bar{u} \in H_0^1(\mathbb{R}_+)} \frac{(\hat{\mathcal{L}}_\sigma \bar{u}, \bar{u})}{(\hat{\mathcal{B}}\bar{u}, \bar{u})} = \frac{(\hat{\mathcal{L}}_\sigma u_\sigma, u_\sigma)}{(\hat{\mathcal{B}}u_\sigma, u_\sigma)}. \quad (23)$$

In particular, for $\sigma = 0$, we have

$$0 < R_{\min}(a, 0) = \inf_{\bar{u} \in H_0^1(\mathbb{R}_+)} \frac{(\hat{\mathcal{L}}_0 \bar{u}, \bar{u})}{(\hat{\mathcal{B}}\bar{u}, \bar{u})} \leq \frac{(\hat{\mathcal{L}}_0 u_\sigma, u_\sigma)}{(\hat{\mathcal{B}}u_\sigma, u_\sigma)} = \frac{(\hat{\mathcal{L}}_\sigma u_\sigma, u_\sigma)}{(\hat{\mathcal{B}}u_\sigma, u_\sigma)} + \sigma \frac{(u_\sigma, u_\sigma)}{(\hat{\mathcal{B}}u_\sigma, u_\sigma)} = R_{\min}(a, \sigma) + \sigma \frac{\|u_\sigma\|_2^2}{(\hat{\mathcal{B}}u_\sigma, u_\sigma)}.$$

This implies

$$R_{\min}(a, \sigma) - R_L(a) \geq \sigma \frac{\|u_\sigma\|_2^2}{-(\hat{\mathcal{B}}u_\sigma, u_\sigma)} =: h(\sigma). \quad (24)$$

In a similar way we find the upper bound

$$R_{\min}(a, \sigma) - R_L(a) \leq \frac{\|u_0\|_2^2}{-(\hat{\mathcal{B}}u_0, u_0)} \sigma =: c\sigma. \quad (25)$$

This proves (22) and in particular $R_{\min}(a, \sigma) \geq R_L(a)$ for $\sigma \geq 0$. Because $(\hat{\mathcal{B}}u_0, u_0) < 0$, inequalities (24) and (25) also imply

$$0 < \frac{\|u_0\|_2^2}{-(\hat{\mathcal{B}}u_0, u_0)} = \frac{\partial R_{\min}(a, \sigma)}{\partial \sigma} \Big|_{\sigma=0}.$$

Differentiating $\hat{\mathcal{L}}_\sigma u_\sigma = R_{\min}(a, \sigma) \hat{\mathcal{B}}u_\sigma$ with respect to σ and setting $v_\sigma = du_\sigma/d\sigma$ gives

$$\hat{\mathcal{L}}_\sigma v_\sigma - u_\sigma = R_{\min}(a, \sigma) \hat{\mathcal{B}}v_\sigma + \frac{\partial R_{\min}(a, \sigma)}{\partial \sigma} \hat{\mathcal{B}}u_\sigma,$$

implying

$$(\hat{\mathcal{L}}_\sigma v_\sigma - R_{\min}(a, \sigma) \hat{\mathcal{B}}v_\sigma, v_\sigma) - (u_\sigma, v_\sigma) = \frac{\partial R_{\min}(a, \sigma)}{\partial \sigma} (\hat{\mathcal{B}}u_\sigma, v_\sigma).$$

Since $\hat{\mathcal{B}}$ is self-adjoint in $L^2(\mathbb{R}_+)$, we have

$$(u_\sigma, v_\sigma) = \frac{1}{2} \frac{d}{d\sigma} \|u_\sigma\|_2^2, \quad (\hat{\mathcal{B}}u_\sigma, v_\sigma) = \frac{1}{2} \frac{d}{d\sigma} (\hat{\mathcal{B}}u_\sigma, u_\sigma),$$

$$(\hat{\mathcal{L}}_\sigma v_\sigma - R_{\min}(a, \sigma) \hat{\mathcal{B}}v_\sigma, v_\sigma) < 0,$$

we thus find

$$\frac{d}{d\sigma} \|u_\sigma\|_2^2 < -\frac{\partial R_{\min}(a, \sigma)}{\partial \sigma} \frac{d}{d\sigma} (\hat{\mathcal{B}}u_\sigma, u_\sigma). \quad (27)$$

Next we define $\tilde{h}(\sigma) := -\|u_\sigma\|_2^2/(\hat{\mathcal{B}}u_\sigma, u_\sigma)$ such that $h(\sigma) = \sigma \tilde{h}(\sigma)$. Differentiating $\tilde{h}(\sigma)$ and using (26) and (27) results in

$$-(\hat{\mathcal{B}}u_0, u_0) \frac{d}{d\sigma} \|u_\sigma\|_2^2 \Big|_{\sigma=0} + \|u_0\|_2^2 \frac{d}{d\sigma} (\hat{\mathcal{B}}u_\sigma, u_\sigma) \Big|_{\sigma=0} < 0,$$

which immediately implies $\tilde{h}'(0) < 0$. Since $h''(\sigma) = 2\tilde{h}'(\sigma) + \sigma \tilde{h}''(\sigma)$, we find $h''(0) < 0$ which proves the second assertion of the theorem. \square

The function $R_{\min}(a, \sigma)$ for $a = 0.759$ together with the upper and lower bounds are shown in Fig. 2(B). Theorem 2 implies the following. Let the system Rayleigh number R_s be sufficiently close to $R_L(a)$. If $R_s > R_L(a)$, then there exists a $\sigma > 0$ such that $R_s = R_{\min}(a, \sigma)$. In other words, if $R_s > R_L(a)$, there exists an infinitesimal perturbation which will grow in time, implying that the boundary layer is unstable. If $R_s < R_L(a)$ no definite statement about stability can be made. Only certain infinitesimal perturbations now decay. Others, and in particular large perturbations, may still grow in time.

4. Transient growth behaviour

In this section we investigate the previous statement further. Employing the energy method, one seeks for conditions in terms of the system parameters (a, R) for which a suitable norm – mostly the L^2 -norm – decays in time. In [3] it is shown that if $a > 0$ and $R < R_E(a)$, see also Fig. 1, then solutions of the linear problem (LP) satisfy

$$\frac{d}{dt} \int_{\mathbb{R}_+} s^2(z, t) dz < 0 \quad \text{for all } t > 0,$$

provided $f \in H_0^1(\mathbb{R}_+)$. Moreover, this statement holds for *periodic* (with respect to the horizontal (x, y) -plane) solutions of the original nonlinear perturbation equations (9).

To exploit the behaviour when $R_E(a) < R < R_L(a)$, we first introduce some notation.

Definition 3. Let $s(z, t)$ be the solution of problem (LP) with $f \in L^2(\mathbb{R}_+)$ and let

$$\mathcal{E}(t) := \int_{\mathbb{R}_+} s^2(z, t) dz =: \|s(t)\|_2^2. \quad (28)$$

(i) The spectral bound of the operator L is defined by

$$\sigma(L) = \sup_{\lambda \in \Sigma(L)} \operatorname{Re} \lambda,$$

where $\Sigma(L)$ denotes the spectrum of L .

(ii) The numerical range of the operator L is the set

$$W(L) = \{(Ls, s) : \|s\|_2 = 1\}.$$

(iii) A related quantity is the numerical bound of L , defined by

$$\omega(L) = \sup_{\lambda \in W(L)} \operatorname{Re} \lambda. \quad (29)$$

(iv) For $a > 0$ and $R > 0$ let $\mathbb{G} := \{(a, R) : \omega(L) > 0, \sigma(L) < 0\}$ denote the stability gap. Clearly, $\mathbb{G} = \{(a, R) : a > 0, R_E(a) < R < R_L(a)\}$.

An important property of the numerical range is that the spectrum of the operator L is contained in the closure of its numerical range: i.e. $\Sigma(L) \subset \overline{W(L)}$, see for instance [26]. This implies that $\omega(L) \geq \sigma(L)$.

From (LP), using Definition 3, we find the relation

$$\frac{1}{2} \mathcal{E}'(t) = \frac{(Ls, s)}{\|s\|_2^2} \mathcal{E}(t) \leq \omega(L) \mathcal{E}(t). \quad (30)$$

The linear problem is stable with respect to $\|\cdot\|_2$ provided $\omega(L) < 0$ and neutral stable when $\omega(L) \equiv 0$. The Euler–Lagrange equation for the maximum problem (30) is given by

$$D^2 s - a^2 s + \frac{1}{2} R \{e^{-z} B(s) + B(e^{-z} s)\} = \omega s, \quad (31)$$

which we solve for $s \in H_0^1(\mathbb{R}_+)$. Let ω_{\max} be the largest real eigenvalue of (31). Then $\omega(L) = \omega_{\max}$. Fig. 3(A) shows the isocurves of ω_{\max} . Observe that the zero-level curve corresponds to $R_E(a)$, see also Fig. 1.

A formal Taylor expansion of $\mathcal{E}(t)$ at $t = 0$ gives

$$\mathcal{E}(t) = \mathcal{E}(0) + \mathcal{E}'(0)t + O(t^2).$$

Now using (30), we have $\mathcal{E}'(0)/\mathcal{E}(0) = 2(Lf, f)/\|f\|_2^2$, where f is the initial condition given by (LP)4. Let μ denote the maximal initial slope of $\mathcal{E}(t)/\mathcal{E}(0)$, i.e.

$$\frac{1}{2} \mu = \sup_{\tilde{f} \in H_0^1(\mathbb{R}_+)} \frac{(L \tilde{f}, \tilde{f})}{\|\tilde{f}\|_2^2} = \omega(L) \quad \text{by Definition 3(ii)–(iii).}$$

The initial condition f that maximizes the initial slope is found by solving the eigenvalue problem

$$\frac{1}{2} (L + L^\dagger) \tilde{f} = \frac{1}{2} \mu \tilde{f} \quad \text{in } H_0^1(\mathbb{R}_+). \quad (32)$$

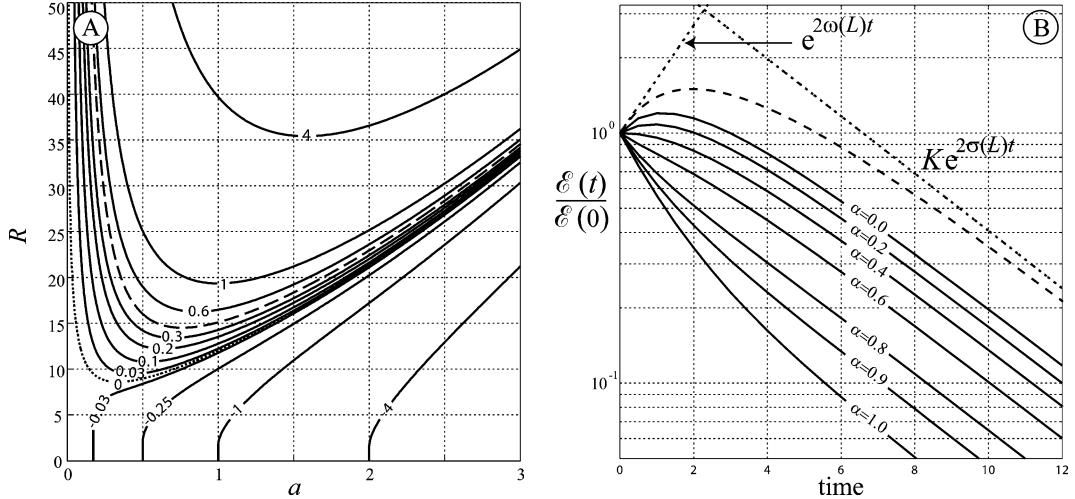


Fig. 3. (A) Isocurves of the largest eigenvalue ω_{\max} of (31). Dashed curve corresponds to $R_L(a)$ and dotted curve to $R_E(a)$ (= zero level curve). (B) Transient growth for $a = 0.38$ and $R = 15$ (in G) of the scaled ‘energy’. For the dashed curve the initial condition is given by $f = \tilde{f}$ (see (LP)). The initial perturbation for the solid curves is given by a specific f for which $e^{\frac{1}{2}z}f \in L^2(\mathbb{R}_+)$. The upper bound (34) is shown as the composition of the dotted lines.

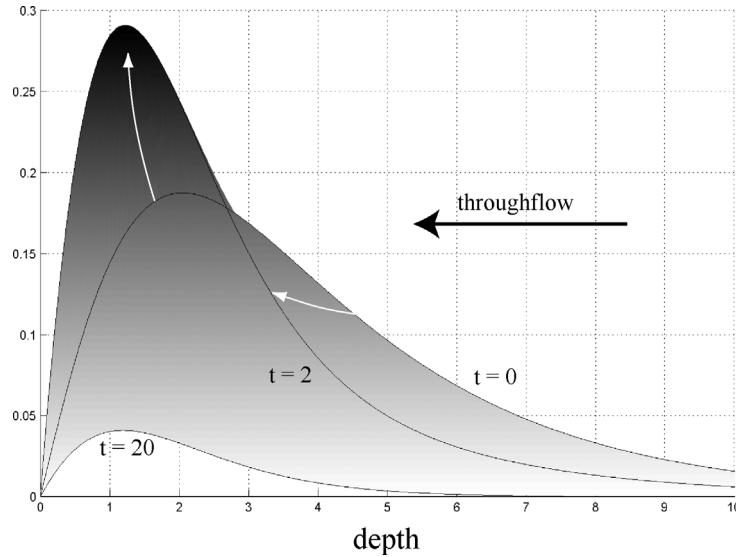


Fig. 4. Evolution of the solution of (LP) with initial perturbation $\tilde{f}(z)$. The corresponding functional \mathcal{E} is depicted in Fig. 3(B) for $\alpha = 0$.

The ‘energy’ $\mathcal{E}(t)$ corresponding to the maximal initial growth $\tilde{f}(z)$ is shown in Fig. 3(B), dashed curve. The computed evolution of the initial perturbation $\tilde{f}(z)$ is shown in Fig. 4. The perturbation ‘mass’ is redistributed due to the existing upward throughflow, implying transient growth of $\mathcal{E}(t)$. Since the system is (linearly) stable, i.e. $\sigma(L) < 0$, the perturbation will eventually decay to zero.

We conclude with some bounds on the growth of $\mathcal{E}(t)$. From (\widehat{LP}) and using Definition 3(i) we obtain

$$\|u(t)\|_2^2 \leq e^{2\sigma(\hat{L})t} \|g\|_2^2. \quad (33)$$

From (33), (16) and using the fact that $\sigma(\hat{L}) = \sigma(L)$, it follows that

$$\|s(t)\|_2^2 \leq \|u(t)\|_2^2 \leq K e^{2\sigma(L)t} \|f\|_2^2,$$

where $K := \|g\|_2^2/\|f\|_2^2$. Further, we have the estimate

$$\|s(t)\|_2^2 \leq e^{2\omega(L)t} \|f\|_2^2,$$

which follows directly from Definition 3. Combining these two estimates results in the upper bound

$$\frac{\mathcal{E}(t)}{\mathcal{E}(0)} \leq \min_{t \geq 0} \{e^{2\omega(L)t}, K e^{2\sigma(L)t}\}, \quad (34)$$

see also the dotted lines in Fig. 3(B). Note that this upperbound is not sharp.

5. Elimination of transient growth by weighted norms

In Section 3 we showed that the linear operator L can be symmetrized by transformation (16). In this section we generalize this idea by defining

$$s(z, t) := e^{-\frac{1}{2}\alpha z} u(z, t), \quad \alpha \in [0, 1]. \quad (35)$$

Substitution of (35) in (LP) gives the problem

$$(LP_\alpha) \quad \begin{cases} \partial_t u = D^2 u + (1 - \alpha) D u + \left(\frac{1}{4}\alpha^2 - \frac{1}{2}\alpha - a^2\right) u \\ \quad + R e^{(\frac{1}{2}\alpha-1)z} B(e^{-\frac{1}{2}\alpha z} u) =: L_\alpha u & \text{in } Q, \\ B^{-1}(w) := -a^{-2} D^2 w + w = e^{-\frac{1}{2}\alpha z} u & \text{in } \mathbb{R}_+, \text{ for each } t > 0, \\ w(0, t) = u(0, t) = 0 & \text{for all } t > 0, \\ u(z, 0) = g(z) & \text{for all } z > 0. \end{cases}$$

Note that $e^{-\frac{1}{2}\alpha z} L_\alpha e^{\frac{1}{2}\alpha z} \equiv L$, i.e. L_α is similar to L for each $\alpha \in [0, 1]$.

From this point on we redefine the energy functional. Let $u(z, t)$ be a solution of (LP_α) . Then

$$\mathcal{E}(t) := \int_{\mathbb{R}_+} u^2(z, t) dz = \int_{\mathbb{R}_+} s^2(z, t) e^{\alpha z} dz,$$

where $s(z, t)$ is the solution of (LP). We want to investigate the transient growth of solutions of (LP_α) for $\alpha \in [0, 1]$. We argue as in Section 4 to achieve monotonic decay of $\mathcal{E}(t)$ with respect to time. The variational formulation based on L_α yields the eigenvalue problem

$$D^2 u + \left(\frac{1}{4}\alpha^2 - \frac{1}{2}\alpha - a^2\right) u = -\frac{1}{2} R \{e^{(\frac{1}{2}\alpha-1)z} B(e^{-\frac{1}{2}\alpha z} u) + e^{-\frac{1}{2}\alpha z} B(e^{(\frac{1}{2}\alpha-1)z} u)\}, \quad (36)$$

for $u \in H_0^1(\mathbb{R}_+)$. For given $a > 0$ and $\alpha \in [0, 1]$, let $R_E(a; \alpha)$ be the smallest positive eigenvalue of (36). Observe that for $\alpha = 1$ we regain eigenvalue problem (21) with $\sigma = 0$, i.e. neutral stability. Therefore, $R_E(a; 1) = R_L(a)$. For $\alpha = 0$, eigenvalue problem (36) reduces to the one discussed in [3] and hence $R_E(a; 0) = R_E(a)$. The stability curves $R = R(a; \alpha)$ for $a > 0$ and for fixed $\alpha \in [0, 1]$ are shown in Fig. 5, see also Fig. 1. They are ordered in the following sense.

Proposition 4. For any $\alpha \in [0, 1)$ we have $R_E(a; \alpha) < R_L(a)$ for all $a > 0$.

Proof. Let $a > 0$ be fixed and let $(s_1, w_1, R_L(a))$ be the first eigensolution of the problem (in \mathbb{R}_+)

$$\begin{cases} D^2 s_1 + D s_1 - a^2 s_1 = -R_L(a) e^{-z} w_1, \end{cases} \quad (37a)$$

$$\begin{cases} -D^2 w_1 + a^2 w_1 = a^2 s_1, \end{cases} \quad (37b)$$

with $s, w \in H_0^1(\mathbb{R}_+)$. Next set $s_1 = e^{-\frac{1}{2}\alpha z} u_1$. Substitution in (37) yields

$$\begin{cases} D^2 u_1 + (1 - \alpha) D u_1 + \left(\frac{1}{4}\alpha^2 - \frac{1}{2}\alpha - a^2\right) u_1 = -R_L(a) e^{(\frac{1}{2}\alpha-1)z} w_1, \end{cases} \quad (38a)$$

$$\begin{cases} -D^2 w_1 + a^2 w_1 = a^2 u_1 e^{-\frac{1}{2}\alpha z}. \end{cases} \quad (38b)$$

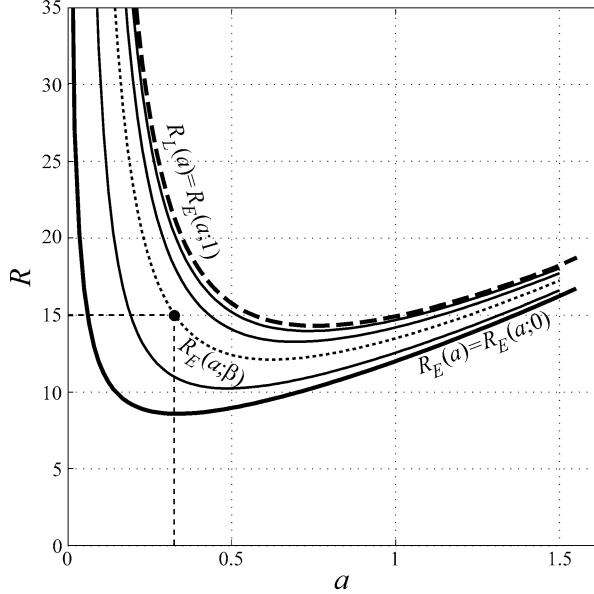


Fig. 5. Lowest eigenvalue $R_E(a; \alpha)$, for various $\alpha \in [0, 1]$, versus wavenumber $a > 0$ for the equilibrium boundary layer (cf. Proposition 4). Bold solid curve ($\alpha = 0$) and bold dashed curve ($\alpha = 1$) are taken from Fig. 1. The dashed curve corresponds to $\alpha = \beta \in (0, 1)$.

Multiplying (38a) by u_1 and integrating over \mathbb{R}_+ gives for $\alpha \in [0, 1)$

$$\frac{1}{R_L(a)} = \frac{(e^{(\frac{1}{2}\alpha-1)z} w_1, u_1)}{\|Du_1\|_2^2 - (\frac{1}{4}\alpha^2 - \frac{1}{2}\alpha - a^2)\|u_1\|_2^2}.$$

From this we immediately derive

$$\frac{1}{R_E(a; \alpha)} > \frac{1}{R_L(a)},$$

for all $a > 0$. \square

The curves in Fig. 5 have the following interpretation. The operator L_α , defined in equation $(LP_\alpha)_1$, is self-adjoint (and hence normal) for $\alpha = 1$. For this particular case we find $R_E(a; 1) = R_L(a)$ for all $a > 0$, see Section 4. For $\alpha \in [0, 1)$ we have $R_E(a; \alpha) < R_L(a)$ for all $a > 0$, i.e. there exists a stability gap. Now let the system Rayleigh number R_s be such that for some $a > 0$ it satisfies $R_E(a) < R_s < R_L(a)$. Then there exists a unique $\beta \in (0, 1)$ such that $R_E(a, \beta) = R_s$, see Fig. 5, dotted curve. Then the described construction implies that for each $\alpha \in (\beta, 1)$ we have

$$\frac{d}{dt} \int_{\mathbb{R}_+} s^2(z, t) e^{\alpha z} dz < 0 \quad \text{provided} \quad \int_{\mathbb{R}_+} f^2(z) e^{\alpha z} dz < \infty.$$

For fixed $(a, R) \in \mathbb{G}$ we solved (LP) . The initial condition f is chosen in such a way that $\int_{\mathbb{R}_+} f^2(z) e^z dz < \infty$. The behaviour of $\mathcal{E}(t)/\mathcal{E}(0)$ is shown in Fig. 3(B). The maximal initial growth of $\mathcal{E}(t)/\mathcal{E}(0)$ is obtained for $\alpha = 0$. For increasing α , the initial slope decreases and becomes negative. This is also to be expected since for some $\beta \in (0, 1)$ the pair (a, R) lies under the stability curve $R_E(a, \beta)$, i.e. the numerical bound becomes negative, see Fig. 5. From this we conclude that $\int_{\mathbb{R}_+} s^2(z, t) e^{\alpha z} dz$ exhibits transient growth for $\alpha \in (0, \beta)$.

6. Discussion

In [3] we introduced and studied gravitational instability of a saline boundary layer formed by evaporation induced upward throughflow at the horizontal surface of a porous medium. In that study several important questions remained untouched. The purpose of the present paper is to resolve these questions. In particular, we present a complete picture of the spectrum of the linearised problem and we prove exchange of stabilities by introducing a similar self-adjoint operator.

Because of the evaporation induced convection, the linearised perturbation equations are non-self-adjoint. This results in the appearance of a stability gap, i.e. the linear instability bound does not coincide with the energy stability bound. By using again similar operators, and thereby introducing weighted norms in the energy method, we show that these bounds coincide for small perturbations.

The stability gap usually suggests the existence of subcritical *finite* amplitude solutions, i.e. nontrivial solutions that exist for Rayleigh numbers below the critical linear instability threshold. The existence of such solutions is shown by Pieters and Schuttelaars, [27]. These authors describe a fairly complete bifurcation analysis of the problem discussed in this paper. The existence of subcritical solutions, however, does not contradict the coinciding stability thresholds since we do *not* incorporate *finite* amplitude nonlinear interactions in our approach. To be more precisely, for finite amplitude steady solutions we generally have $(\mathcal{N}(s), s)\mathcal{M} \neq 0$.

References

- [1] R.A. Wooding, S.W. Tyler, I. White, Convection in groundwater below an evaporating salt lake: 1. Onset of instability, *Water Resour. Res.* 33 (1997) 1199–1217.
- [2] R.A. Wooding, S.W. Tyler, I. White, P.A. Anderson, Convection in groundwater below an evaporating salt lake: 2. Evolution of fingers or plumes, *Water Resour. Res.* 33 (6) (1997) 1219–1228.
- [3] C.J. van Duijn, G.J.M. Pieters, R.A. Wooding, A. van der Ploeg, Stability criteria for the vertical boundary layer formed by throughflow near the surface of a porous medium, in: P.A.C. Raats, D. Smiles, A.W. Warrick (Eds.), *Environmental Mechanics – Water, Mass and Energy Transfer in the Biosphere – The Philip Volume*, in: *Geophys. Monographs*, vol. 129, American Geophysical Union, 2002, pp. 155–169.
- [4] S.H. Davis, Buoyancy, surface tension in stability by the method of energy, *J. Fluid Mech.* 39 (1969) 347.
- [5] S.H. Davis, On the principle of exchange of instabilities, *Proc. Roy. Soc. Ser. A* 310 (1969) 341–358.
- [6] S.H. Davis, On the possibility of subcritical instabilities, in: *Instability of Continuous Systems*, Proc. IUTAM Sympos. Herrenalb, Springer-Verlag, Berlin, 1971, pp. 222–227.
- [7] G.M. Homsy, A.E. Sherwood, Convective instabilities in porous media with throughflow, Technical Report UCRL-76539, Lawrence Livermore Lab. Rep., 1975.
- [8] G.M. Homsy, A.E. Sherwood, Convective instabilities in porous media with throughflow, *Amer. Inst. Chem. Engrs. J.* 22 (1976) 168–174.
- [9] G.P. Galdi, B. Straughan, Exchange of stabilities, symmetry, and nonlinear stability, *Arch. Rational Mech. Anal.* 89 (1985) 211–228.
- [10] G.P. Galdi, M. Padula, A new approach to energy theory in the stability of fluid motion, *Arch. Rational Mech. Anal.* 110 (3) (1990) 187–286.
- [11] B. Straughan, *The Energy Method, Stability and Nonlinear Convection*, second ed., *Appl. Math. Sci.*, vol. 91, Springer-Verlag, New York, 2003.
- [12] C.J. van Duijn, G.J.M. Pieters, P.A.C. Raats, Steady flows in unsaturated soils are stable, *Trans. Porous Media* 57 (2004) 215–244.
- [13] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, New York, 1976.
- [14] S.C. Reddy, D.S. Henningson, Energy growth in viscous channel flows, *J. Fluid Mech.* 252 (1993) 209–238.
- [15] B.F. Farrell, P.J. Ioannou, Generalized stability theory. Part I: Autonomous operators, *J. Atmos. Sci.* 53 (14) (1996) 2025–2040.
- [16] C.E. Grosch, H. Salwen, The continuous spectrum of the Orr–Sommerfeld equation. Part 1. The spectrum and the eigenfunctions, *J. Fluid Mech.* 87 (1978) 33–54.
- [17] H. Salwen, C.E. Grosch, The continuous spectrum of the Orr–Sommerfeld equation. Part 2. Eigenfunction expansions, *J. Fluid Mech.* 104 (1981) 445–465.
- [18] D.G. Lasseigne, R.D. Joslin, T.L. Jackson, W.O. Criminale, The transient period for boundary layer disturbances, *J. Fluid Mech.* 381 (1999) 89–119.
- [19] W.O. Criminale, T.L. Jackson, R.D. Joslin, *Theory and Computation of Hydrodynamic Stability*, Cambridge University Press, Cambridge, 2003.
- [20] P.J. Schmidt, D.S. Henningson, *Stability and Transition in Shear Flows*, Springer, Berlin, 2001.
- [21] D.A. Nield, A. Bejan, *Convection in Porous Media*, second ed., Springer-Verlag, New York, 1999.
- [22] G.J.M. Pieters, I.S. Pop, C.J. van Duijn, A note on the solution of a coupled parabolic-elliptic system arising in linear stability analysis of gravity-driven porous media flow, Technical Report CASA 04-30, Eindhoven University of Technology, 2004, submitted for publication to the ANZIAM J.
- [23] E.R. Lapwood, Convection of a fluid in a porous medium, *Proc. Cambridge Phil. Soc.* 44 (1948) 508–521.
- [24] G.J.M. Pieters, I.S. Pop, A modified Chebyshev–Galerkin method for parabolic problems on unbounded domains, submitted for publication.
- [25] S.C. Reddy, L.N. Trefethen, Pseudospectra of the convection–diffusion operator, *SIAM J. Appl. Math.* 54 (1994) 1634–1649.
- [26] V. Istratescu, *Introduction to Linear Operators*, Dekker, New York, 1981.
- [27] G.J.M. Pieters, H.M. Schuttelaars, Bifurcation analysis for the steady boundary layer formed by throughflow near the surface of a porous medium, submitted for publication.